However, the inviscid method is still accurate to within 10% if the wedge angles are greater than 8° and $M_0 < 16$. The trend of the empirical solution for $M_0 > 12$ shows that its maximum deviation from the viscous solution for ϵ occurs at the minimum values of δ_{b_1}, with the intermediate values of δ_{b_1} yielding the minimum differences between the two solutions.

It was also found that the variation of ϵ between the inviscid and viscous solution was a maximum for the combination of high wedge body temperature, low total pressure, high Mach number, and shock interaction location near the leading edge. The effect of total temperature was found to be negligible.

A more complete analysis of the data obtained in the comparison of the three methods of solution may be found in Ref. 7.

Experimental Results

The experimental portion of this investigation was conducted in the 20-in. hypersonic wind tunnel located in the Aerospace Research Laboratories at Wright-Patterson Air Force Base, Ohio. A complete description of the facility and its calibration data may be found in Ref. 8.

The tests to obtain this shock interaction data were conducted at a nominal freestream Mach number of 14.36 and total pressure and total temperature conditions of 1600 psia and 1850°R, respectively. The freestream Reynolds number per foot was 0.729×10^6. The range of wedge body angles used in the experimental program included angles from 11.95° to 27.33°.

Since there is a density gradient across the contact surface, it should be visible to schlieren observation. However, because of the heat conduction between the permanently adjacent particles on either side of the discontinuity, the contact surface layer will tend to dissipate rather rapidly. Thus, it was considered improbable at the outset of the tests that this region of discontinuity would be visible. Subsequent testing verified this assumption, although in a few cases this surface was visible and measurable. However, since the shock angles were visible, this information could be used to find δ_3 and δ_4 from Eq. (1) or (2). However, because of possible errors in measuring the shock wave angles, it is evident that the values of ϵ computed may not exactly agree. In general, however, it was found that the values of ϵ agreed to measurable accuracy.

Using the measured values of θ_3 and θ_4 to compute ϵ, it was found that the average error between this value and the value of ϵ from the viscous solution was 7.1%, with the maximum error encountered being 34.3%. In the cases where the contact surface was visible to schlieren observation, the average error between the viscous solution for ϵ and the measured value was 7.5%, with the maximum error being 19.4%.

Conclusions

In general, it was found that the inviscid solution is an accurate method for predicting the contact surface angle for all wedge angles at hypersonic Mach numbers below 10. Above Mach 10, however, some care must be exercised in using the inviscid solution, especially for small wedge angles. For values of wedge body angles above 8°, however, the inviscid solution is fairly accurate up to a Mach number of 16. Beyond Mach 16, it is not recommended to use the inviscid solution for any values of wedge angles.

The empirical equations developed to predict ϵ are highly accurate for Mach numbers below 10, considering the vast amount of calculations that they save. At high Mach numbers, however, care should be exercised in their use, just as for the inviscid solution. The empirical equations are, however, slightly more accurate for the higher Mach numbers than the inviscid solution.

Analyzation of the experimental data shows that the viscous solution for predicting the angle of the contact surface with respect to the freestream flow agrees quite well with the experimental data for a Mach number of 14.36. The errors that are present may well be attributed to inaccuracies in measuring the shock wave angles. Hence, it would appear from these results that the viscous method as developed in this note for predicting ϵ is valid for any hypersonic Mach number, at least up to the value used in the experimental portion of these tests.

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Addendum: "The Vertical Water-Exit and -Entry of Slender Symmetric Bodies"

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Nomenclature

C_F = upward thrust coefficient (net upward force/maximum frontal area/freestream dynamic pressure)
Fr = square of Froude number (U^2/gl)
g = acceleration due to gravity
l = body length
$S(y)$ = body cross-sectional area, $\pi X^2(y)$†
U = body speed
$X(y)$ = body radius†
x = radial coordinate†
y = axial coordinate,† measured downward from upper end of body
Z = displacement of upper end of body above undisturbed position of free surface†
ϕ = velocity potential, made dimensionless with Ul
τ = body thickness ratio, maximum diameter/length

Introduction

THE purpose of this note is to remove certain limitations of the solution derived in Ref. 1, viz., its inapplicability to blunt-ended bodies of revolution and its restriction to large but finite Froude number situations.

Treatment of Blunt-Ended Bodies of Revolution

When applied to round-ended bodies, such as the ellipsoid of revolution, the formal slender-body theory of Ref. 1 pre-

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† X, x, y, and Z are dimensionless, normalized by the body length l.

dicts infinite axial forces during the broach (partial submergence) phase of the motion. The source of this anomaly is that part of the body-surface pressure distribution which is the same as if the flow were unbounded. That is, this contribution is too strongly infinite near round stagnation points to be integrated.

This deficiency can be removed by a more exact treatment of the body boundary condition, which was formally linearized in Ref. 1. If the ends of the body are parabolic (blunt with finite radius of curvature), and if the body cross-sectional area distribution is expandable in power series about the stagnation points that converge over the length of the body, the theory of Ref. 1 can be rendered a valid first approximation near the body's extremities simply by introducing gaps between these extremities and the ends of the body-bound source distribution. Specifically, the distribution should be terminated at points midway between the ends of the body and the centers of curvature of the extremities[2] [see Eq. (2)].

For bodies not meeting these requirements, the singularities probably should not be confined to the body axis but should be distributed on the body surface. A treatment of water-exit and -entry along the lines of Ref. 1 is then still possible in principle. The strength of the body-bound singularity distribution could be taken to be the same as if the flow were unbounded, with errors no more (nor less) serious than those incurred in making the same approximation in Ref. 1. Of course, considerable recourse to high-speed computers would be necessary, even for the determination of the singularity strength.[3, 4]

Froude-Number Effects

As noted in Ref. 5, the analysis of Ref. 1 is applicable to water-entry problems only in the infinite Froude number approximation.‡ Even in the exit situation, the analysis of Ref. 1 is valid only for large (but finite) Froude numbers, since it is based on expansions in inverse powers of the Froude number. Recently, the solution for the vertical constant-speed motion of a constant-strength source toward a free surface has been put in terms of source distributions.[7] This permits extension of the analysis of Ref. 1 to the vertical water-exit of slender symmetric bodies at arbitrary Froude number. For the case where the body is completely submerged, the potential may be obtained by straightforward superposition of the solution of Ref. 7. In the nomenclature of Ref. 1, we have, for the axisymmetric problem,

$$\phi_T = -\frac{1}{4\pi}\int_\alpha^\beta S'(\eta)\{x^2 + (y-\eta)^2\}^{-1/2}d\eta + \frac{1}{4\pi}\int_\alpha^\beta S'(\eta)\{x^2 + (y+\eta-2Z)^2\}^{-1/2}d\eta - \frac{1}{2\pi}\frac{1}{Fr}\int_\alpha^\beta d\xi\int_\xi^\infty d\eta\, \exp\left\{\frac{(\xi-\eta)}{Fr}\right\} \times S'(\xi)\,\{x^2 + (y+\eta-2Z)^2\}^{-1/2} \quad (1)$$

Here α and β may be set equal to 0 and 1, respectively (i.e., the sources may be allowed to extend to the ends of the body) when the ends of the body are pointed, without incurring serious errors.[8] For blunt-ended bodies meeting the restrictions just delineated (parabolic ends, etc.),

$$\alpha = (1/4\pi)S'(0) \qquad \beta = 1 + (1/4\pi)S'(1) \quad (2)$$

In this latter case, it can be shown that the solution given by Eqs. (1) and (2) is a uniformly valid first approximation when the depth of submergence is of the order of a body length.[2]

To extend the solution to the broach phase of the motion, we adopt the simplifying assumption, made in Ref. 1 and elsewhere,[9, 10] that the only effect of the free surface on the body-bound singularity distribution is to extinguish the sources as they cross the surface. Then, as a first approximation to the solution, simply let $\alpha \to Z$ in Eq. (1), where $y = Z$ is the undisturbed free surface. The resultant potential, $\phi_T^{(0)}$, say, does not satisfy the free-surface boundary condition [Eq. (4) of Ref. 1]. An additional term, $\phi_T^{(1)}$, say, must be found, essentially to account for the destruction of the sources as they cross the surface. We determine $\phi_T^{(1)}$ by expanding it and the free-surface boundary condition in inverse powers of the Froude number. The general term of the series may be determined and the series summed to yield

$$\phi_{T^{(1)}} = -\frac{1}{2\pi}\frac{1}{Fr}\{x^2 + (y-Z)^2\}^{-1/2} \times \int_0^Z \exp\left\{\frac{(\xi-Z)}{Fr}\right\} S(\xi)d\xi - \frac{1}{2\pi}\frac{1}{Fr}\int_0^Z \exp\left\{\frac{(\xi-Z)}{Fr}\right\} S'(\xi)d\xi \times \int_0^\infty \exp\left(\frac{-\eta}{Fr}\right)\{x^2 + (y+\eta-Z)^2\}^{-1/2}d\eta \quad (3)$$

The results of Ref. 1 do not quite agree with the asymptotic expansion in Fr^{-1} of the present formulas; the singularity distributions in Eqs. (32, 33, 35, and 37) of Ref. 1 which extend from $y = Z$ to $y = 2Z - 1$ (from the free surface to the image point of the tail of the body) should run instead from $(2Z - 1)$ to $-\infty$ [cf. the discussion following Eq. (32) of Ref. 1]. However, the difference is a singularity distribution of time-independent strength and location in space-fixed coordinates. This does not affect the pressures and forces on the body in the slender-body approximation, so that those results of Ref. 1 are not invalidated by the present analysis.

Calculations based on the present Eq. (1) have been made of the axial force felt by a slender ellipsoid of revolution in vertical motion toward the surface at Froude numbers from zero to infinity. The results, shown in Fig. 1, reinforce the conclusion of Ref. 1 that Froude number effects are negligible for practical values of this parameter insofar as axial forces are concerned. Similarly, an extension of the solution just discussed to the lifting problem shows that, as claimed in Ref. 1, the lateral forces and pitching moments experienced by a vertically exiting body are independent of Froude number.

These conclusions regarding Froude number effects must be qualified by the neglect of cavitation effects here and in Ref. 1. This remains a major defect of water-exit and -entry theory. Also, the solution discussed herein is only formally correct near the intersection of the body axis and the free surface when the body is close to or is broaching the surface. Analyses have been made in which the body boundary condition is satisfied more precisely during the broach phase of

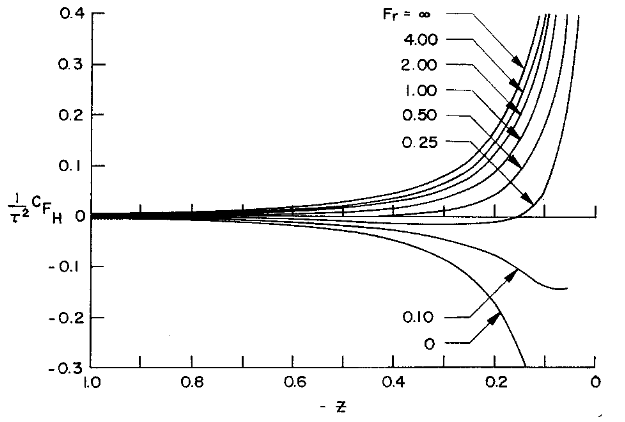

Fig. 1 Hydrodynamic contribution to upward thrust (net axial force minus buoyant force) on slender ellipsoid of revolution in vertical water-exit.

‡ An improved proof of the equivalence of exit and entry problems in this approximation is contained in Ref. 6.

the motion,[11] but, since the free-surface boundary conditions used were still formally linearized, it is uncertain as to whether or not such procedures are a step in the right direction. A completely satisfactory answer to this question would seem to require a wholy numerical solution.

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Motion of the Center of Gravity of a Variable-Mass Body

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Nomenclature

B = body of variable mass moving in space
O' = a point fixed in space
O = a point fixed in the body B (origin of body-axes system)
G = center of gravity of B; G changes its position with respect to O as the mass varies
dm = element of mass of body B located at a point P fixed in the body
$\mathbf{R}$ = O′P
$\mathbf{r}$ = OP
$\mathbf{R}^{(O)}$ = O′O
$\mathbf{R}^{(G)}$ = O′G
$\mathbf{r}^{(G)}$ = OG
$\mathbf{F}$ = external force acting on the body
$\mathbf{K}$ = reactive force acting on the body, produced by the mass ejection
ω = angular velocity of the body B
d/dt = derivative with respect to a fixed point O_1
$\delta/\delta t$ = derivative with respect to a point O, moving with the body
M = $\int dm$, the total mass of the body B at the moment under consideration

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MASS is continuously ejected from some portion on the surface of body B. Mass is ejected with a nonzero velocity relative to the point O, and consequently the reactive forces are produced. It is assumed that ejected mass, after its separation from the body, does not affect in any way the motion of the body.

Because of the mass ejection, the center of gravity G is displaced relative to the point O. The objective of this paper is to derive the equation of motion for the center of gravity G.

The equation of motion for the body B can be expressed in the following form:

$$\int (d^2\mathbf{R}/dt^2)dm = \mathbf{F} + \mathbf{K} \tag{1}$$

where the integration is extended over the mass of the body at the time t.

For any arbitrary point P of the body B, we can write

$$d^2\mathbf{R}/dt^2 = (d^2\mathbf{R}^{(0)}/dt^2) + \omega \times \mathbf{r} + \omega \times (\omega \times \mathbf{r}) \tag{2}$$

or, integrating over the mass of the body B, we can write

$$\int (d^2\mathbf{R}/dt^2)dm = \int (d^2\mathbf{R}^{(0)}/dt^2)dm + \int \omega \times \mathbf{r}dm + \int \omega \times (\omega \times \mathbf{r})dm \tag{3}$$

Since

$$\int \mathbf{r}dm = M\mathbf{r}^{(G)} \tag{4}$$

Eq. (3) can be written in the form

$$\int (d^2\mathbf{R}/dt^2)dm = M[(d^2\mathbf{R}^{(0)}/dt^2) + \dot{\omega} \times \mathbf{r}^{(G)} + \omega \times (\omega \times \mathbf{r}^{(G)})] \tag{5}$$

Since G is not fixed in the body B,

$$d^2\mathbf{R}^{(G)}/dt^2 = (d^2\mathbf{R}^{(0)}/dt^2) + \dot{\omega} \times \mathbf{r}^{(G)} + \omega \times (\omega \times \mathbf{r}^{(G)}) + 2\omega \times (\delta\mathbf{r}^{(G)}/\delta t) + (\delta^2\mathbf{r}^{(G)}/\delta t^2) \tag{6}$$

Combining Eqs. (1, 5, and 6), we can write

$$M\frac{d^2\mathbf{R}^{(G)}}{dt^2} = \mathbf{F} + \mathbf{K} + 2M\omega \times \frac{\delta r^{(G)}}{\delta t} + M\frac{\delta^2\mathbf{r}^{(G)}}{\delta t^2} \tag{7}$$

which represents the equation of motion for the variable center of gravity of the variable-mass body.

Flutter Characteristics of Titanium Alloys

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Nomenclature

E = modulus of elasticity
I = second moment of area
K = constant
M = bending moment
V = velocity
b = one-half of the chord at reference station
c = distance from neutral axis to outer fiber of section
w = weight
μ = mass ratio
σ = bending stress
ψ = flutter parameter ratio
ω = circular frequency

Introduction

MUCH has been done to perfect materials whose performance characteristics can meet the demands of high-speed flight at temperatures well above the functional range of

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